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SPREADING OUT OF A VISCOUS LIQUID OVER A HORIZONTAL SURFACE

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The spreading out of a viscous liquid over the surface of a solid body plays an important role in a number of practical problems, for example, in the formation of coatings of solid bodies, in the motion of gas–liquid mixtures and emulsions in capillaries, and in other cases [1]. The motion of a thin film of a viscous liquid over a horizontal surface is caused by the action of gravity and surface tension forces and has much in common with the motion of thin films over an inclined surface, which has been intensively studied for a number of years [2-4]. The transition from an inclined plane to a horizontal one is not trivial, i.e., it does not reduce to the substitution into the final formulas of a slope angle equal to zero. The point is that motion over a horizontal surface is described even in the crudest approximation by a differential equation of higher order.

The problem of the spreading out of a viscous liquid over a horizontal surface has been discussed in the two-dimensional formulation in [5], in which the approximate nonlinear equation for the layer thickness h is obtained as a function of the coordinate x and the time t:

$$h_t = (g/3\nu)(h^3 h_x)_x. \tag{1}$$

Here ν is the kinematic viscosity coefficient and g is the gravitational acceleration. Unfortunately, the effect of surface tension has in fact not been taken into account in [5].

In the opposite limiting case, in which one can neglect the force of gravity in comparison with the surface tension force, the equation for h(x, t) has been obtained in [6] (also only in the two-dimensional formulation):

$$h_t + (\sigma/3\rho\nu)(h^3 h_{xxx})_x = 0, \tag{2}$$

where σ is the surface tension coefficient and ρ is the density of the liquid.

The three-dimensional problem of the motion of a viscous incompressible liquid over a horizontal plane is discussed in this paper with account taken of the gravity and surface tension forces. The slope of the free surface is assumed to be small, and the motion is assumed to be sufficiently slow (creeping) so that one can neglect the inertial forces in comparison with the viscous ones. As will be shown, the Reynolds number does not necessarily have to be small. No restrictions are imposed on variations of the layer thickness h(x, y, t); in particular, h can vanish, as occurs upon the spreading out of a drop.

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We shall utilize the Navier—Stokes equations with the following boundary conditions for the description of the dynamics of the liquid. There is adhesion on the surface of the solid body:

$$\mathbf{u}|_{z=0} = 0, \quad w|_{z=0} = 0. \tag{3}$$

Here $\mathbf{u}$ is the horizontal velocity vector, w is the vertical component of the velocity, and z is the vertical coordinate.

On the free surface continuity of the normal σ_{nn} and tangential $\sigma_{n\tau}$ stresses is the boundary condition [7]:

$$\sigma_{nn}|_{z=h} = -p_a + \sigma\frac{\Delta h}{[1+(\nabla h)^2]^{3/2}}, \quad \sigma_{n\tau}|_{z=h} = 0, \tag{4}$$

where p_a is the atmospheric pressure (we shall assume that p_a = const), σ is the surface tension coefficient, and ∇ and Δ are the two-dimensional (in the horizontal x, y plane) Hamiltonian and Laplacian operators. It is convenient to combine the kinematic boundary condition with the continuity equation and write it in the form

$$h_t + \nabla\int_0^h \mathbf{u}\,dz = 0. \tag{5}$$

Let the characteristic thickness of the layer be equal to H, and let the characteristic horizontal dimension be L. We shall select $U = gH^3/\nu L$, which is determined from the condition that the characteristic values of the horizontal gradient of the hydrostatic pressure and the viscosity forces balance each other, as the scale of the horizontal velocity. Let us introduce dimensionless variables by the formulas:

$$\mathbf{u} = U\mathbf{u}', \quad w = \alpha U w', \; x = Lx', \; y = Ly',$$
$$z = H\xi, \; h = H\eta, \; t = T\tau, \; p = p_a + \rho g(h-z) + P\,p',$$

where

$$\alpha = H/L; \; P = \rho g H; \; T = L/U = \nu L^2/gH^3,$$

the hydrostatic component $\rho g(h-z)$ of the pressure p is separated out as a separate term for convenience.

In these variables the Navier—Stokes equations, the incompressibility condition, and Eq. (5) are written in the form

$$\mathrm{Fr}^2\left(\mathbf{u}'_\tau + (\mathbf{u}'\nabla)\,\mathbf{u}' + w'\mathbf{u}'_\xi\right) = -\nabla\eta - \nabla p' + \mathbf{u}'_{\xi\xi} + \alpha^2\Delta\mathbf{u}'; \tag{6}$$

$$\alpha^2\,\mathrm{Fr}^2\left(w'_\tau + (\mathbf{u}'\nabla)\,w' + w'w'_\xi\right) = -\,p'_\xi + \alpha^2 w'_{\xi\xi} + \alpha^4\Delta w'; \tag{7}$$

$$\Delta\mathbf{u}' + w'_\xi = 0; \tag{8}$$

$$\eta_\tau + \nabla\int_0^\eta \mathbf{u}'\,d\xi = 0, \tag{9}$$

where $\mathrm{Fr}^2 = gH^5/\nu^2L^2 = U^2/gH$ is the square of the Froude number. For brevity's sake the dimensionless operators ∇ and Δ are denoted by the same symbols as are the dimensional operators.

Let us introduce an equation which describes the evolution of the free surface $\eta(x', y', \tau)$ for small values of α^2 and sufficiently slow motions of the liquid. It is evident from Eqs. (6) and (7) that in the problem under discussion the smallness of the Froude number Fr and not the Reynolds number $\mathrm{Re} = UH/\nu = gH^4/\nu^2 L$ is the "slowness" condition of the motion, i.e., smallness of the inertial terms in comparison with the viscous ones. It is easy to convince oneself that the Reynolds number is related to the Froude number by the relationship $\mathrm{Fr}^2 = \alpha\,\mathrm{Re}$, so that the condition $\mathrm{Fr}^2 \ll 1$ can be satisfied for $\mathrm{Re} \gg 1$ only if $\alpha^2 \ll 1$. Consequently, it is convenient in the asymptotic analysis to make use of the expansion of the quantities u', w', and p' into series in the two independent small parameters α^2 and Fr^2:

$$\mathbf{u}' = \sum_{m,n} \mathbf{u}'_{mn} \alpha^{2m} \mathrm{Fr}^{2n}. \tag{10}$$

A closed equation for η is obtained as follows. We substitute the expansion (10) into Eqs. (6)-(8) and the boundary conditions and solve this problem, assuming $\eta(x', y', \tau)$ to be specified. In each approximation in α^2 and Fr^2 the solution reduces to the integration of known functions vertically and can be performed in final form. The solution found $\mathbf{u}' = \mathbf{u}'_{00} + \mathrm{Fr}^2\mathbf{u}'_{01} + \alpha^2\mathbf{u}'_{10} + \ldots$, which is functionally dependent on η, is then substituted into Eq. (9). It is interesting that with this choice it is not necessary to decompose the quantity η into a series in small parameters. This not only simplifies the calculations but also permits not imposing any additional restrictions on the amplitude of the variations of η.

We note that in thin liquid films the relation $\alpha^2 \ll \mathrm{Fr}^2 \ll 1$ is often satisfied. For example, for water with H = 0.3 mm, L = 1 cm, and $\nu = 10^{-2}$ cm²/sec we obtain $\alpha^2 = 9 \times 10^{-4}$ and $\mathrm{Fr}^2 = 0.2 \gg \alpha^2$. Therefore in the asymptotic expansion (10) it is necessary first of all to take account of corrections in Fr^2, one can completely ignore terms $\sim \alpha^2$ in the lowest approximations (which are of main interest). The boundary conditions (4) simplify significantly and take the form

$$\mathbf{u}'_{\xi}|_{\xi=\eta} = 0,\ p'|_{\xi=\eta} = -\mathrm{We}\Delta\eta, \tag{11}$$

in the dimensionless variables, where $\mathrm{We} = \sigma/\rho g L^2$ is the Weber number.

In the zeroth approximation in α^2 and Fr^2 the horizontal velocity vector $\mathbf{u}'_{00}$ found from (3), (6), (7), and (11) is of the form

$$\mathbf{u}'_{00} = \eta^2\left(\frac{\xi^2}{2\eta^2} - \frac{\xi}{\eta}\right)\nabla(\eta - \mathrm{We}\Delta\eta). \tag{12}$$

Substituting this expression into (9), we obtain a nonlinear equation for $\eta(x', y', \tau)$ of the zeroth approximation.* We write it immediately in dimensional variables:

$$h_t - \frac{g}{3\nu}\nabla(h^3\nabla h) + \frac{\sigma}{3\rho\nu}\nabla(h^3\nabla\Delta h) = 0. \tag{13}$$

After $\eta(x', y', \tau)$ is found from the solution of Eq. (13), we find the horizontal velocity by formula (12) and the vertical velocity from the incompressibility condition (8).

In the first approximation in Fr^2 the equation for η appears more cumbersome; therefore we write it out for the case in which one can neglect the term with We:

$$\eta_t - \frac{1}{3}\nabla(\eta^3\nabla\eta) + \mathrm{Fr}^2\nabla\left\{\frac{47}{180}\eta^6\nabla\eta(\nabla\eta)^2 + \frac{641}{3780}\eta^7\Delta\eta\nabla\eta + \frac{4}{105}\eta^7\nabla[(\nabla\eta)^2] + \frac{2}{45}\eta^8\Delta\nabla\eta\right\} = 0.$$

Let us analyze in detail the equation of the zeroth approximation (13). In the two limiting cases this equation changes into the well-known equations. When $\sigma = 0$ (surface tension is absent) and $\partial/\partial y = 0$, we obtain Eq. (1). When $g = 0$ (there is no gravity) and $\partial/\partial y = 0$, we obtain Eq. (2).

When $\sigma = 0$, Eq. (13) belongs to the class of nonlinear parabolic equations, numerous self-similar solutions of which have been found in [9, 10] in connection with problems of the theory of thermal waves and filter theory. In particular, if when an axisymmetric liquid drop spreads out there is an inflow of liquid at its center with intensity Q and the condition $\eta = 0$ is satisfied at infinity, the self-similar solution of the contracted Eq. (13) has the form indicated in [10] and is different from zero only in the finite region $0 < r < R(t)$. The radius of the drop R increases with time according to the law

$$R(t) = AQ^{3/8}(g/\nu)^{1/8}t^{1/2}. \tag{14}$$

Numerical integration gives the value 0.62 for the coefficient A. Formula (14) is in agreement (to within an accuracy of some difference in a numerical coefficient) with the experi-

*For the particular case $\partial/\partial y = 0$ this equation is obtained by a different method in [8], which a reviewer has kindly pointed out to the author.

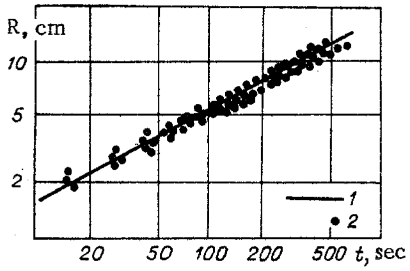

Fig. 1

mental results of [11], in which an axisymmetric glycerin spot spread out on glass. The theoretical dependence (14) with A = 0.3 is denoted in Fig. 1 by the numeral "1", and the experimental data of [11] with Q = 0.91 ± 0.04 cm^3/sec and ν = 6.2 cm^2/sec are denoted by the numeral "2". Estimates show that under the conditions of these tests We << 1, i.e., the effect of surface tension is actually small. If there is no inflow of liquid, the self-similar solution of Eq. (13) gives an expansion according to the law

$$R = \text{const}\ V^{3/8}(g/\nu)^{1/8}t^{1/8}. \tag{15}$$

for an axisymmetric drop of volume V.

When $\sigma \neq 0$, Eq. (13) contains spatial derivatives of higher order, and additional boundary conditions, for example, the value of the wetting angle, are necessary for its solution.

In the simplest case of a two-dimensional ($\partial/\partial y = 0$) motionless drop of volume V it is easy to obtain from Eq. (13) the shape h(x) and length 2d of the drop analytically:

$$h(x) = \beta(\text{ch}\ ad - \text{ch}\ ax)/(a\ \text{sh}\ ad),$$

where $a = \sqrt{\rho g/\sigma}$ is the capillary constant, β is the tangent of the wetting angle, and the quantity d is determined from the equation $(1 + Va^2/2\beta)\tanh ad = ad$, which is in complete agreement with the known result from hydrostatics [12].

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